4-7 Identity and Inverse Matrices

Main Ideas

- Determine whether two matrices are inverses.
- Find the inverse of a 2×2 matrix.

New Vocabulary

identity matrix
inverse

GET READY for the Lesson

With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using a "key."

The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message must multiply by the inverse of the coding matrix.

Code																	
_	0	A	1	B	2	C	3	D	4	E	5	F	6	G	7	H	8
I	9	J	10	K	11	L	12	M	13	N	14	O	15	P	16	Q	17
R	18	S	19	T	20	U	21	V	22	W	23	X	24	Y	25	Z	26

Identity and Inverse Matrices Recall that for real numbers, the multiplicative identity is 1. For matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix.

2×2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

KEY CONCEPT — Identity Matrix for Multiplication

Word The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.

Symbols If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

EXAMPLE Verify Inverse Matrices

1 Determine whether each pair of matrices are inverses of each other.

a. $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ **and** $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

If X and Y are inverses, then $X \cdot Y = Y \cdot X = I$.

$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$ Write an equation.

$= \begin{bmatrix} 1 - 2 & 1 + \frac{1}{2} \\ -\frac{1}{2} + (-4) & -\frac{1}{2} + 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ Matrix multiplication

Since $X \cdot Y \neq I$, they are *not* inverses.

Study Tip

Verifying Inverses

Since multiplication of matrices is not commutative, it is necessary to check the product in both orders.

b. $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

If P and Q are inverses, then $P \cdot Q = Q \cdot P = I$.

$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$ Write an equation.

$= \begin{bmatrix} 3 - 2 & -6 + 6 \\ 1 - 1 & -2 + 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Matrix multiplication

$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ Write an equation.

$= \begin{bmatrix} 3 - 2 & 4 - 4 \\ -\frac{3}{2} + \frac{3}{2} & -2 + 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Matrix multiplication

Since $P \cdot Q = Q \cdot P = I$, P and Q are inverses.

CHECK Your Progress

1. $X = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

Find Inverse Matrices Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

KEY CONCEPT *Inverse of a 2 × 2 Matrix*

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Notice that $ad - bc$ is the value of det A. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

EXAMPLE Find the Inverse of a Matrix

2 **Find the inverse of each matrix, if it exists.**

a. $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

First find the determinant to see if the matrix has an inverse.

$$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$$

Since the determinant equals 0, R^{-1} does not exist.

b. $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{3(2) - 1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$ $a = 3, b = 1, c = 5, d = 2$

$$= 1\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$ Simplify.

CHECK Find the product of the matrices. If the product is I, then they are inverses.

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 2 - 2 \\ -15 + 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

CHECK Your Progress

2A. $\begin{bmatrix} -3 & 7 \\ 1 & -4 \end{bmatrix}$

2B. $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$

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Matrices can be used to code messages by placing the message in a $n \times 2$ matrix.

Real-World Link

The Enigma was a German coding machine used in World War II. Its code was considered to be unbreakable. However, the code was eventually solved by a group of Polish mathematicians.

Source: bletchleypark.org.uk

Study Tip

Messages

If there is an odd number of letters to be coded, add a 0 at the end of the message.

Real-World EXAMPLE

3 **a. CRYPTOGRAPHY Use the table at the beginning of the lesson to assign a number to each letter in the message GO_TONIGHT. Then code the message with the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.**

Convert the message to numbers using the table.

G O _ T O N I G H T

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

Write the message in matrix form. Arrange the numbers in a matrix with 2 columns and as many rows as are needed. Then multiply the message matrix B by the coding matrix A.

$$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 14 + 60 & 7 + 45 \\ 0 + 80 & 0 + 60 \\ 30 + 56 & 15 + 42 \\ 18 + 28 & 9 + 21 \\ 16 + 80 & 8 + 60 \end{bmatrix}$$ Multiply the matrices.

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix}$$ Write an equation.

The coded message is 74 | 52 | 80 | 60 | 86 | 57 | 46 | 30 | 96 | 68.

b. Use the inverse matrix A^{-1} to decode the message in Example 3a.

First find the inverse matrix of $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{2(3) - (1)(4)}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$ $a = 2, b = 1, c = 4, d = 3$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$ Simplify.

(continued on the next page)

Next, decode the message by multiplying the coded matrix C by A^{-1}.

$$CA^{-1} = \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 111 - 104 & -37 + 52 \\ 120 - 120 & -40 + 60 \\ 129 - 114 & -43 + 57 \\ 69 - 60 & -23 + 30 \\ 144 - 136 & -48 + 68 \end{bmatrix}$$ Multiply the matrices.

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix}$$ Simplify.

Use the table again to convert the numbers to letters. You can now read the message.

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

G O _ T O N I G H T

CHECK Your Progress

3. Use the table at the beginning of the lesson to assign a number to each letter in the message SECRET_CODE. Then code the message with the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Use the inverse matrix A^{-1} to decode the message.

CHECK Your Understanding

Example 1 (p. 209)

Determine whether each pair of matrices are inverses of each other.

1. $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$

2. $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

3. $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

4. $F = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, G = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

Example 2 (p. 210)

Find the inverse of each matrix, if it exists.

5. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$

Example 3 (pp. 211–212)

8. **CRYPTOGRAPHY** Code a message using your own coding matrix. Give your message and the matrix to a friend to decode. (*Hint*: Use a coding matrix whose determinant is 1 and that has all positive elements.)

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–12	1
13–21	2
22–24	3

Determine whether each pair of matrices are inverses of each other.

9. $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

10. $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$

12. $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

13. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

16. $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

19. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

20. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

21. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

CRYPTOGRAPHY For Exercises 22–24, use the alphabet table at the right.
Your friend sent you messages that were coded with the coding matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the inverse of matrix C to decode each message.

CODE					
A	26	J	17	S	8
B	25	K	16	T	7
C	24	L	15	U	6
D	23	M	14	V	5
E	22	N	13	W	4
F	21	O	12	X	3
G	20	P	11	Y	2
H	19	Q	10	Z	1
I	18	R	9	_	0

22. 50 | 36 | 51 | 29 | 18 | 18 | 26 | 13 | 33 | 26 | 44 | 22 | 48 | 33 | 59 | 34 | 61 | 35 | 4 | 2

23. 59 | 33 | 8 | 8 | 39 | 21 | 7 | 7 | 56 | 37 | 25 | 16 | 4 | 2

24. 59 | 34 | 49 | 31 | 40 | 20 | 16 | 14 | 21 | 15 | 25 | 25 | 36 | 24 | 32 | 16

25. **RESEARCH** Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded.

Determine whether each statement is *true* or *false*.

26. Only square matrices have multiplicative identities.

27. Only square matrices have multiplicative inverses.

28. Some square matrices do not have multiplicative inverses.

29. Some square matrices do not have multiplicative identities.

Determine whether each pair of matrices are inverses of each other.

30. $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

31. $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

EXTRA PRACTICE
See pages 899, 929.
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Find the inverse of each matrix, if it exists.

32. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

33. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

34. $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

35. GEOMETRY Compare the matrix used to reflect a figure over the x-axis to the matrix used to reflect a figure over the y-axis.

a. Are they inverses?

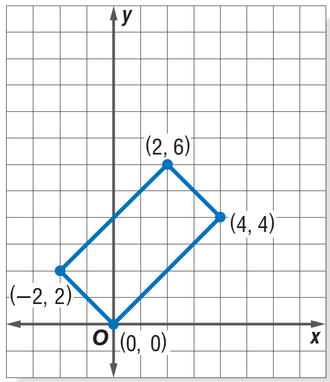
b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

36. GEOMETRY The matrix used to rotate a figure 270° counterclockwise about the origin is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Compare this matrix with the matrix used to rotate a figure 90° counterclockwise about the origin.

a. Are they inverses?

b. Does your answer make sense? Use a drawing to support your answer.

GEOMETRY For Exercises 37–41, use the figure at the right.



37. Write the vertex matrix A for the rectangle.

38. Use matrix multiplication to find BA if $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

39. Graph the vertices of the transformed rectangle. Describe the transformation.

40. Make a conjecture about what transformation B^{-1} describes on a coordinate plane.

41. Find B^{-1} and multiply it by BA. Make a drawing to verify your conjecture.

Graphing Calculator

INVERSE FUNCTION The x^{-1} key on a TI-83/84 Plus graphing calculator is used to find the inverse of a matrix. If you get a SINGULAR MATRIX error on the screen, then the matrix has no inverse. Find the inverse of each matrix.

42. $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$

43. $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$

44. $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$

H.O.T. Problems

45. REASONING Explain how to find the inverse of a 2 × 2 matrix.

46. OPEN ENDED Create a square matrix that does not have an inverse. Explain how you know it has no inverse.

47. CHALLENGE For which values of a, b, c, and d will $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$?

48. *Writing in Math* Use the information about cryptography on page 208 to explain how inverse matrices are used in cryptography. Explain why the inverse matrix works in decoding a message, and describe the conditions you must consider when writing a message in matrix form.

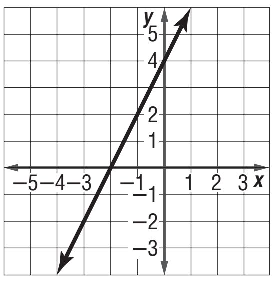
STANDARDIZED TEST PRACTICE

49. ACT/SAT The message **MEET_ME_ TOMORROW** is converted into numbers (0 = space, A = 1, B = 2, etc.) and encoded using a numeric key. After the message is encoded it becomes 31 | −11 | 30 | 50 | 13 | 39 | 10 | −10 | 55 | 5 | 41 | 19 | 54 | 18 | 53 | 39. Which key was used to encode this message?

A $\begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$

B $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

50. REVIEW Line q is shown below. Which equation best represents a line parallel to line q?



F $y = x + 2$ **H** $y = 2x - 3$

G $y = x + 5$ **J** $y = -2x + 2$

Spiral Review

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

51. $3x + 2y = -2$
$x - 3y = 14$

52. $2x + 5y = 35$
$7x - 4y = -28$

53. $4x - 3z = -23$
$-2x - 5y + z = -9$
$y - z = 3$

Evaluate each determinant. (Lesson 4-5)

54. $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$

55. $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$

56. $\begin{vmatrix} 5 & -7 & 3 \\ -1 & -2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$

Find each product, if possible. (Lesson 4-3)

57. $[5 \quad 2] \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

58. $\begin{bmatrix} 7 & 4 \\ -1 & 2 \\ -3 & 5 \end{bmatrix} \cdot [3 \quad 5]$

59. $[4 \quad 2 \quad 0] \cdot \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$

Solve each system of equations. (Lesson 3-2)

60. $3x + 5y = 2$
$2x - y = -3$

61. $6x + 2y = 22$
$3x + 7y = 41$

62. $3x - 2y = -2$
$4x + 7y = 65$

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

63. (2, 5), (6, 9)
64. (1, 0), (−2, 9)
65. (−5, 4), (−3, −6)
66. (−2, 2), (−5, 1)
67. (0, 3), (−2, −2)
68. (−8, 9), (0, 6)

69. OCEANOGRAPHY The bottom of the Mariana Trench in the Pacific Ocean is 6.8 miles below sea level. Water pressure in the ocean is represented by the function $f(x) = 1.15x$, where x is the depth in miles and $f(x)$ is the pressure in tons per square inch. Find the pressure in the Mariana Trench. (Lesson 2-1)

GET READY for the Next Lesson

Solve each equation. (Lesson 1-3)

70. $3k + 8 = 5$
71. $12 = -5h + 2$
72. $7z - 4 = 5z + 8$
73. $\frac{x}{2} + 5 = 7$
74. $\frac{3 + n}{6} = -4$
75. $6 = \frac{s - 8}{-7}$